TOPOLOGY - III, EXERCISE SHEET 7

Exercise 1. *Homology of a wedge sum.*
Let $\{(X_\alpha, x_\alpha)\}_{\alpha\in I}$ be pointed spaces such that there exist open neighbourhoods $U_\alpha \subseteq X_\alpha$ which contract to x_α. Show that we have isomorphisms of reduced homology groups

$$\tilde{H}_i(\bigvee_{\alpha\in I} X_\alpha) \cong \bigoplus_{\alpha\in I} \tilde{H}_i(X_\alpha)$$

for all $i > 0$. Recall that the space $\bigvee_{\alpha\in I} X_\alpha$ is constructed by gluing the spaces X_α such that the base points x_α are identified.

Exercise 2. *Relative homology isn't always homology of quotient.*
Give an example of a space X and a subspace $A \subseteq X$ such that $H_i(X, A) \ncong H_i(X/A)$ for some $i \geq 0$.

Exercise 3. *Brouwer fixed point theorem.*
Let $f : D^n \to D^n$ be a continuous map from the closed disc in $\mathbb{R}^n$ to itself. Show that there exists a point $p \in D^n$ such that $f(p) = p$.
Hint: If f does not have a fixed point then show that one can construct a retraction $D^n \to \partial D^n \cong S^{n-1}$. Then using exercises 2.(1) of sheet 5 and 2.(2) of sheet 6, show that this is not possible.

Exercise 4. $\mathbb{R}^n \cong \mathbb{R}^m \implies n = m$.
The goal of this exercise is to show that given open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ such that U and V are homeomorphic, then $n = m$.

(1) Let $x \in U$. Show that $\tilde{H}_i(U, U - \{x\}) \cong \tilde{H}_i(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ using excision.
(2) Using the long exact sequence of relative homology show that $\tilde{H}_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \tilde{H}_{i-1}(\mathbb{R}^n - \{x\})$.
(3) Compute $\tilde{H}_{i-1}(\mathbb{R}^n - \{x\})$ for all i and conclude.

Exercise 5. Go through and understand the proof of the excision theorem from the notes by Rich Schwartz.